Forward and Inverse Kinematics of Three Degrees of Freedom Spherical Parallel Manipulators

ED5314
Course Project

submitted by
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1 Introduction

Spherical parallel robots are special kind of spatial robots that provide three degrees of freedom of pure rotation. The links of SPMs move on an imaginary sphere(s) in space. The platform vertices and the links move on spheres with coincident centers All the rotation axes pass through this imaginary center point.

Three legged non redundant SPMs have two links in each of the legs. Moving platform and the links may be joint either by spherical or revolute joints. Rotary actuators have been used for actuation. If we calculate degrees of freedom of 3$\underline{R}$RRSPM we observe that it comes out to be -3! The calculation is as follows- Six links and the platform each have 6-DOF in space. This sums up to 42-DOF. Each leg has three revolute joints- the actuated joint, the proximal-distal link joint and the platform- distal link joint, each allowing only 1-DOF of rotary motion. Therefore, we have total of 45-DOF constrained! However the fact that 3$\underline{R}$RRSPM under discussion is symmetrical in nature, sorts out this issue. If we replace the platform-distal link revolute joints by spherical joints (which restrict only 3-DOF in space), then we correctly predict the degrees of freedom of the platform.

Before we start analyzing forward and inverse kinematics of 3$\underline{R}$RRSPM, we prepare ourselves by studying the behaviour of a simple 2R spherical serial robot. The forward and inverse kinematic analysis of 2R spherical serial manipulator is carried out in section 2.Then we move to the inverse kinematic problem of 3$\underline{R}$RRSPM in section 3.1. Solution to the forward kinematics problem using joint space approach and task space approach is discussed in section 3.2.1 and 3.2.2 respectively.

Literature discussing kinematics of 3$\underline{R}$RRSPM is little compared to other spatial 3-DOF manipulators.It is reported that there exist maximum of eight solutions to forward kinematics problem of 3-DOF SPMs [2]. [3] uses spherical analytical theory (spherical trigonometry) heavily for derivations. [2] follows task space approach with rotation matrix assumed in terms of Euler angles for forward kinematic analysis. The body fixed frames and Euler angles are chosen smartly to simplify the equations. [1] uses input output equations of spatial four bar mechanism to write loop closure equations. The equations are solved semi-graphically. The paper also reviews the literature in area of SPMs nicely.

In the joint space approach, angle of rotation of motor and angle between proximal and distal links are chosen as variables. This results in total of six variables, out of which the rotation of motor is specified leaving three unknowns. Three equations constructed based on geometry of the platform, lead to a univariate polynomial in half tangent of one of the unknown angles (hence forth will be called FKU), when solved by successive elimination.

In task space approach, orientation of the moving platform is assumed in terms of unit quaternion. Task space equations form a set of four variable quadratic polynomials. The system is simplified by moving to canonical coordinate systems. The polynomial system is reduced to two equations and two unknowns. Calculation of resultant as well as Bezoutain leads probably to a very high degree FKU. Complete calculations were not possible due limited computational resources. A search for simplification of equations is continued.

2 Forward and inverse kinematics of 2R spherical serial

Let the point of concurrency all the revolute joint axes be the origin $\boldsymbol{O}$. Link 1 and link 2 subtain angles α_1 and α_2 respectively at $\boldsymbol{O}$. The angles α_1 and α_2 are analogous to the link lengths of 2R planar robot.

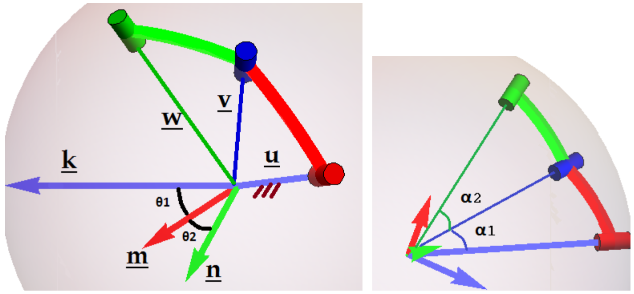

Figure 1: A single link on sphere, and 2R spherical serial

2.1 Forward kinematics

The fixed axis $\boldsymbol{u}$ is specified. Vector $\boldsymbol{k}$, normal to $\boldsymbol{u}$ and the plane of link 1 's initial position is also specified as a frame of reference for measuring the angle θ_1. We use body fixed angles for analysis. So, the angle between link 1 and link 2 θ_2 is also given. We have to find out coordinates of point P.

$$\boldsymbol{m} = Rk\phi(\boldsymbol{u}, \theta_1) \cdot \boldsymbol{k} \tag{1}$$
$$\boldsymbol{v} = Rk\phi(\boldsymbol{u}, \alpha_1) \cdot \boldsymbol{u} \tag{2}$$
$$\boldsymbol{n} = Rk\phi(\boldsymbol{v}, \theta_2) \cdot \boldsymbol{m} \tag{3}$$
$$\boldsymbol{w} = Rk\phi(\boldsymbol{n}, \alpha_2) \cdot \boldsymbol{v} \tag{4}$$

Where
$Rk\phi(\boldsymbol{k}, \phi)$ performs space fixed rotation about axis $\boldsymbol{k}$ with angle ϕ.
$\boldsymbol{v}$ is the position vector of point P_1 and
$\boldsymbol{w}$ is the position vector of point P.

2.2 Inverse kinematics

The $\boldsymbol{u}$ and $\boldsymbol{k}$ from above problem are specified. The angles subtained by link at origin α_1 and α_2 are also given. Along with this information, the coordinates of point P i.e. axis $\boldsymbol{w}$ is also given. We have to find out possible values of $\boldsymbol{v}$ and then θ_1 and θ_2. Let $\boldsymbol{v} = (x, y, z)^T$

$$\boldsymbol{v} \cdot \boldsymbol{w} = \cos(\alpha_1) \tag{5}$$
$$\boldsymbol{v} \cdot \boldsymbol{u} = \cos(\alpha_2) \tag{6}$$
$$\boldsymbol{v} \cdot \boldsymbol{v} = 1 \tag{7}$$

Two of the x, y, z are solved in terms of the third from Eq (5,6) . The solution, on substituting in Eq (7) gives rise to a quadratic equation. Thus, as expected from geometry, we get two solutions for $\boldsymbol{v}$. For each value of $\boldsymbol{v}$ we can calculate θ_1 and θ_2.

$$\boldsymbol{m} = \boldsymbol{u} \times \boldsymbol{v} \tag{8}$$
$$\boldsymbol{n} = \boldsymbol{v} \times \boldsymbol{w} \tag{9}$$
$$\cos(\theta_1) = \boldsymbol{k} \cdot \boldsymbol{m} \tag{10}$$
$$\cos(\theta_2) = \boldsymbol{m} \cdot \boldsymbol{n} \tag{11}$$

Along with $ArcCos()$ while calculating θ_1 and θ_2 from Eq(10,11), we need to prepend proper sign.So,

$$\theta_1 = \text{Sign}((\boldsymbol{k} \times \boldsymbol{m}) \cdot \boldsymbol{u}) \arccos(\boldsymbol{k} \cdot \boldsymbol{m}) \tag{12}$$

$$\theta_2 = \text{Sign}((\boldsymbol{m} \times \boldsymbol{n}) \cdot \boldsymbol{u}) \arccos(\boldsymbol{m} \cdot \boldsymbol{n}) \tag{13}$$

3 Forward and Inverse kinematics of 3$\underline{\text{R}}$RRSPM

The manipulator core structure can be thought of two pyramids (tetrahedrons) pivoted to each other at their apex points. This point is considered as origin $\boldsymbol{O}$. The upper and lower pyramids have apex angles β and γ respectively. The space fixed coordinate system $\Sigma 0$ is set up as shown in Fig.(3). Moving coordinate system $\Sigma 1$ shares origin $\boldsymbol{O}$ with $\Sigma 0$ and has same orientation as $\Sigma 0$ in home position of manipulator, but it is fixed to the upper pyramid. The bottom pyramid is fixed while the top pyramid can rotate freely about $\boldsymbol{O}$. The base of top pyramid can be considered as the manipulator platform.

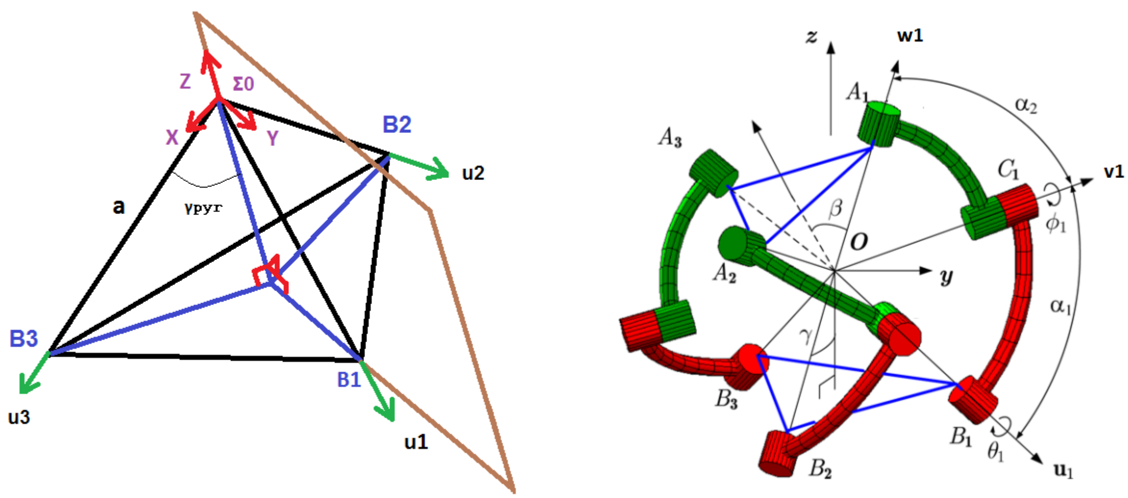

Figure 2: Notation. Figure adopted from (Shaoping Bai et al. 2009)

3.1 Inverse kinematics

The transformation from coordinate system $\Sigma 1$ and $\Sigma 0$ is the input for inverse kinematics. The position vectors of platform vertices A_1, A_2 and A_3 in coordinate system $\Sigma 1$ fixed to it are $\boldsymbol{a_1}, \boldsymbol{a_2}$ and $\boldsymbol{a_3}$ respectively. The position vectors w_1, w_2 and w_3 in $\Sigma 0$ coordinate system are calculated with input rotation matrix ${}^{\boldsymbol{0}}_{\boldsymbol{1}}\boldsymbol{R}$

$$\boldsymbol{w_i} = {}^{\boldsymbol{0}}_{\boldsymbol{1}}\boldsymbol{R} \cdot \boldsymbol{a_i} \quad i = 1, 2, 3 \tag{14}$$

We have three decoupled equations, one for each leg. The inverse kinematics developed for 2R spherical serial can be applied directly to solve this problem by replacing $\boldsymbol{u}$ by $\boldsymbol{u_i}$ and $\boldsymbol{w}$ by $\boldsymbol{w_i}$ in Eq.(5-13). There are two solutions for a leg leading to total of eight solutions.

3.2 Forward kinematics

In the same setup of 3RRRSPM, now three actuator rotation values θ_1, θ_2 and θ_3 are given and the position of the platform i.e. the position vectors A_1, A_2 and A_3 in $\Sigma 0$ frame have to be calculated. The geometry of platform dictates that-

$$\boldsymbol{w_i} \cdot \boldsymbol{w_j} = \cos(\psi) \qquad i, j = 1, 2, 3 \;\; i \neq j \tag{15}$$

Where ψ is the face angle at the apex of the upper pyramid and $\boldsymbol{w_i}$ is position vector of A_i. The rigidity of distal links of the manipulator is expressed in mathematical form as

$$\boldsymbol{v_i} \cdot \boldsymbol{w_i} = \cos(\alpha_2) \qquad i = 1, 2, 3 \tag{16}$$

Where $\boldsymbol{v_i}$ are the intermediate joint (joint connecting distal and proximal links) axis direction cosines.

3.2.1 Joint space approach

The position vectors of vertices of the platform are calculated in terms of the actuated and other joint angles.

$$\boldsymbol{w_i} = \boldsymbol{w_i}(\theta_i, \phi_i) \qquad i = 1, 2, 3 \tag{17}$$

Where ϕ_i is the angle between proximal and distal link of a manipulator leg. On substituting Eq(17) in Eq(15) we get three bilinear equations in sin and cos of ϕ_i with monomials as illustrated.

$$\begin{aligned} &eq\phi_1\phi 2: \\ &\{1, \sin(\phi 2), \sin(\phi 1), \sin(\phi 1)\sin(\phi 2), \cos(\phi 2), \sin(\phi 1)\cos(\phi 2), \cos(\phi 1), \cos(\phi 1)\sin(\phi 2), \cos(\phi 1)\cos(\phi 2) \end{aligned} \tag{18}$$

We can calculate say $\cos(\phi_2)$ and $\sin\phi_2$ from $eq\phi_1\phi_2$ and $eq\phi_2\phi_3$ and then use identity $\cos(\phi_2)^2 + \sin(\phi_2)^2 = 1$ to eliminate ϕ_2. Now we have two equations and two unknowns ϕ_1 and ϕ_3. The equation after elimination of ϕ_2 has form

$$Eq\phi_1\phi_3 = \Sigma_{i=0}^{2}\Sigma_{j=0}^{2} c_{ij}\cos(\phi_1)^i \sin(\phi_1)^{2-i}\cos(\phi_3)^j \sin(\phi_3)^{2-j} \tag{19}$$

At this point, we do half tangent substitution and convert $\cos(\phi_3)$ and $\sin(\phi_3)$ to $\tan(\frac{\phi_3}{2})$ in both the equations $Eq\phi_1\phi_3$ and $eq\phi_1\phi_3$. These two form a set of quartic and quadratic polynomials in $\tan(\frac{\phi_3}{2})$. We perform polynomial division to solve these equations further and get equation only in one unknown ϕ_1.We transform $\cos(\phi_1)$ and $\sin(\phi_1)$ to $\tan(\frac{\phi_1}{2})$ to get FKU.

After successive eliminations, 22 degree FKU is obtained. Full symbolic FKU could not be calculated due to limitation on computational resources. It is observed that the solution method generates six extraneous roots (probably generated from multiplications of $(1 + t^2)$ factors three times) leaving us with 16 solutions satisfying Eq.(15,16). Further it is found that eight of the solutions are reflections of the other eight about the origin $\boldsymbol{O}$. The mathematical relation between such roots is

$$t_i \times t_i^r = -1 \tag{20}$$

Where t_i^r is the root corresponding to the reflected solution of t_i. So, we have eight independent solutions to forward kinematic problem of 3$\underline{\text{R}}$RRSPM. Thus the theorem established in [2] is verified. Author does not claim that he has proved for every case because the calculations of FKU were not completely done symbolically due to limited computing resources.

3.2.2 Task space approach

Orientation of $\Sigma 1$ with respect to $\Sigma 0$ is assumed in terms of unit quaternion $q = (x_0, x_1, x_2, x_3)$. The rotation matrix looks like-

$$ {}^{\mathbf{0}}_{\mathbf{1}}\boldsymbol{R} = \begin{pmatrix} \mathrm{x0}^2 + \mathrm{x1}^2 - \mathrm{x2}^2 - \mathrm{x3}^2 & 2(\mathrm{x1x2} - \mathrm{x0x3}) & 2(\mathrm{x0x2} + \mathrm{x1x3}) \\ 2(\mathrm{x1x2} + \mathrm{x0x3}) & \mathrm{x0}^2 - \mathrm{x1}^2 + \mathrm{x2}^2 - \mathrm{x3}^2 & 2(\mathrm{x2x3} - \mathrm{x0x1}) \\ 2(\mathrm{x1x3} - \mathrm{x0x2}) & 2(\mathrm{x0x1} + \mathrm{x2x3}) & \mathrm{x0}^2 - \mathrm{x1}^2 - \mathrm{x2}^2 + \mathrm{x3}^2 \end{pmatrix} \tag{21} $$

At this point of time we note the fact that the diagonal terms in matrix are squared ones and off-diagonals are bilinear. The position vectors of A_1, A_2 and A_3 are known in $\Sigma 1$. Let $\boldsymbol{A_1}, \boldsymbol{A_2}$ and $\boldsymbol{A_3}$ be the position vectors of points A_1, A_2 and A_3 in $\Sigma 0$ frame. Here we use the fact that the distal links are rigid. It is expressed through following three equations

$$ \boldsymbol{A_i} \cdot \boldsymbol{v_i} = \cos(\alpha_2) \quad i = 1, 2, 3 \tag{22} $$

Along with above equations we have one more constraint equation for unit quaternion.

$$ x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1 \tag{23} $$

Eq(22) and Eq(23) together for set of four quadratic equations in four unknowns x_0, x_1, x_2, x_3. The system has monomials as illustrated-

$$ \text{Unit quaternion constraint :} \ \left\{1, \mathrm{x3}^2, \mathrm{x2}^2, \mathrm{x1}^2, \mathrm{x0}^2\right\} \tag{24} $$

$$ \boldsymbol{A_i} \cdot \boldsymbol{v_i} \ : \ \left\{1, \mathrm{x3}^2, \mathrm{x2x3}, \mathrm{x2}^2, \mathrm{x1x3}, \mathrm{x1x2}, \mathrm{x1}^2, \mathrm{x0x3}, \mathrm{x0x2}, \mathrm{x0x1}\right\} \quad i = 1, 2, 3 \tag{25} $$

This system if reduced to two polynomials in x_1 and x_3, leads to a univariant of degree no less than 164. We realign the fixed and moving reference frames to simplify the equations (24). We understand that Eq.(22) is output of a dot product. The $\boldsymbol{v_i}$s calculated from θ_is are inputs, we assume that at least two of them are independent say $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$. We construct a new basis as follows-

$$ \boldsymbol{Y_0'} = \boldsymbol{v_1} \tag{26} $$

$$ \boldsymbol{Z_0'} = \frac{\boldsymbol{v_1} \times \boldsymbol{v_2}}{||\boldsymbol{v_1} \times \boldsymbol{v_2}||} \tag{27} $$

$$ \boldsymbol{X_0'} = \boldsymbol{Y_0'} \times \boldsymbol{Z_0'} \tag{28} $$

We transform $\boldsymbol{v_i}$s to the new coordinate system $\Sigma 01$. They look like-

$$ \boldsymbol{v_1'} = \{0, 1, 0\} \tag{29} $$

$$ \boldsymbol{v_2'} = \{a, b, 0\} \tag{30} $$

$$ \boldsymbol{v_3'} = \{d, e, f\} \tag{31} $$

Where a-f are just representative symbols. Now, we can clearly see that dot products $\boldsymbol{A_i}.\boldsymbol{v_i'}$ greatly simplify. Now we realign the moving coordinate system $\Sigma 1$ to simplify the $\boldsymbol{A_i}$s. An $\boldsymbol{A_i}$ is result of matrix multiplication of ${}^{\mathbf{0}}_{\mathbf{1}}\boldsymbol{R}$ (Eq.21) and $\boldsymbol{a_i}$. We now have specific requirements. The dot product $\boldsymbol{A_1} \cdot \boldsymbol{v_1'}$ is simplified due to zero x and z coordinate of $\boldsymbol{v_1'}$. Naturally, we should now simplify the y coordinate of $\boldsymbol{A_1}$, which in turn is generated by multiplication of second row of ${}^{\mathbf{0}}_{\mathbf{1}}\boldsymbol{R}$ and $\boldsymbol{a_1}$. In order to eliminate the squared terms, we put a zero at y coordinate of $\boldsymbol{a_1}$ leaving us with two choices of aligning it to new X axis or Z axis. Now, $\boldsymbol{v_2'}$ has a zero at z coordinate, so we need not worry about z coordinate of $\boldsymbol{A_2}$ and consequently $\boldsymbol{a_2}$. So, from this analysis we align new X axis to $\boldsymbol{a_1}$ and align XY plane to the plane of $\boldsymbol{a_1}, \boldsymbol{a_2}$. Our new coordinate moving coordinate system $\Sigma 11$ is setup as follows-

$$\boldsymbol{X_1'} = \boldsymbol{a_1} \tag{32}$$

$$\boldsymbol{Y_1'} = \frac{\boldsymbol{a_1} \times \boldsymbol{a_2}}{||\boldsymbol{a_1} \times \boldsymbol{a_2}||} \tag{33}$$

$$\boldsymbol{Z_1'} = \boldsymbol{X_1'} \times \boldsymbol{Y_1'} \tag{34}$$

We transform $\boldsymbol{a_i}$s to the new coordinate system Σ11. They look like-

$$\boldsymbol{a_1'} = \{1, 0, 0\} \tag{35}$$

$$\boldsymbol{a_2'} = \{p, 0, q\} \tag{36}$$

$$\boldsymbol{a_3'} = \{r, s, t\} \tag{37}$$

So the new task space equations after some simplification have monomials as illustrated below-

$$\text{Unit quaternion constraint}: \ \left\{1, \text{x3}^2, \text{x2}^2, \text{x1}^2, \text{x0}^2\right\} \tag{38}$$

$$\boldsymbol{A_1'} \cdot \boldsymbol{v_1'} \ : \{\text{x2x3}, \text{x0x1}\} \tag{39}$$

$$\boldsymbol{A_2'} \cdot \boldsymbol{v_2'} \ : \left\{1, \text{x3}^2, \text{x1x3}, \text{x1x2}, \text{x1}^2, \text{x0x3}, \text{x0x2}\right\} \tag{40}$$

$$\boldsymbol{A_3'} \cdot \boldsymbol{v_3'} \ : \left\{1, \text{x3}^2, \text{x2}^2, \text{x1x3}, \text{x1x2}, \text{x1}^2, \text{x0x3}, \text{x0x2}, \text{x0x1}\right\} \tag{41}$$

These frame alignments seem to be canonical for the problem and no more initial simplification is expected. We further eliminate x3^2 from Eq.(41). Eq.(39) is used to eliminate x0 which leads to cases with x1 zero or non zero. Polynomial division is performed in variable x1. However, the last two polynomials in two variables still remain complicated and degree of FKU is still very large and the analysis is incomplete.

3.2.3 Numerical example

Following numerical values are chosen to coincide with example given in [1]-
$\alpha_1 = 90^o, \alpha_2 = 45^o, \beta pyr = 60^o, \gamma pyr = 45^o, \theta_1 = 105^o, \theta_2 = 60^o, \theta_3 = 105^o$ The solutions match with the paper.

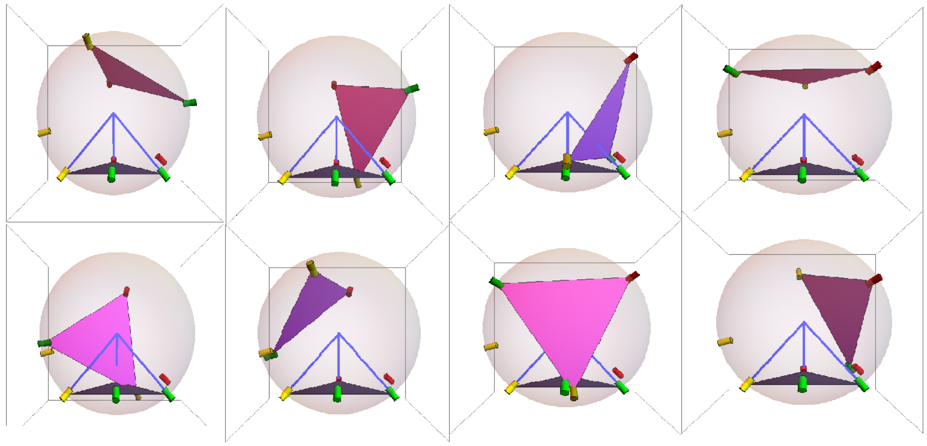

Figure 3: Valid Solutions to Equations (15,16)

4 Conclusions

The methods explored and findings of the forward and inverse kinematics analyses are summarized as follows

- The study of forward and inverse kinematics of 2R spherical serial robot is not found separately in literature as one can find for 2R planar. It has been a part of kinematics of SPMs but separate characterization was not found anywhere, which is done in this project.
- This study uses only one unknown variable per leg of manipulator in joint space approach of forward kinematics. This kind of approach is not specifically seen in literature.
- There exist 16 solutions to forward kinematic analysis of 3$\underline{R}$RRSPM. Only eight out of 16 solutions are independent. The remaining eight solutions i.e. position vectors of moving platform, are reflections of the eight independent solutions about origin.
- Although the task space analysis is not completed yet, approach with rotation matrix assumed in terms of quaternions is not seen in literature of 3$\underline{R}$RRSPM.

A Programs

A few prominent modules written in `Mathematica` during this study

- FwdKin1Rsph[$\boldsymbol{u}$, $\boldsymbol{k}$, α, θ]
- FwdKin2Rsph[$\boldsymbol{u}$, $\boldsymbol{k}$, α_1, α_2, θ_1, θ_2]
- InvKin2Rsph[$\boldsymbol{w}$, $\boldsymbol{u}$, $\boldsymbol{k}$, α_1, α_2]
- BezoutMatrix[c, d]

Where c an d are coefficient arrays of polynomials with the length of coefficient array of the larger polynomial. Other notation is similar to what used in report.

References

[1] Shaoping Bai, Michael R Hansen, and Jorge Angeles. A robust forward-displacement analysis of spherical parallel robots. *Mechanism and Machine Theory*, 44(12):2204–2216, 2009.

[2] C.M. Gosselin, J. Sefrioui, and M.J. Richard. On the direct kinematics of spherical three-degree-of-freedom parallel manipulators with a coplanar platform. *Journal of Mechanical Design, Transactions Of the ASME*, 116(2):587–593, 1994.

[3] Z. Huang and Y.L. Yao. A new closed-form kinematics of the generalized 3-DOF spherical parallel manipulator. *Robotica*, 17:475–485, 1999.